SOLUTION OF A DRYING PROBLEM FOR A COLLOIDAL POROUS MATERIAL WITH A MOVING EVAPORATION BOUNDARY

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Closed solutions are derived for linear differential equations for one-dimensional drying with an evaporation boundary moving in accordance with $\beta\sqrt{t}$.

1. Lykov [1] has given an analytical formulation for the drying in this case with a moving evaporation boundary. He also gave an approximate solution for the simplified case of drying with a linearly moving evaporation boundary.

We use Lykov's analytical scheme to formulate the one-dimensional case of drying with a moving evaporation boundary ($x = y(t)$) for a semiinfinite space; the critical water content U_c is considered as constant at the boundary of the evaporation zone.

The problem is then that of solving a system of equations for the evaporation zone

$$\frac{\partial\theta_1}{\partial t} = \alpha_{11}\frac{\partial^2\theta_1}{\partial x^2} + \alpha_{12}\frac{\partial^2 U_1}{\partial x^2}, \quad 0 < x < y(t), \tag{1.1}$$

$$\frac{\partial U_1}{\partial t} = \alpha_{21}\frac{\partial^2 U_1}{\partial x^2} + \alpha_{22}\frac{\partial^2\theta_1}{\partial x^2}, \quad 0 < x < y(t) \tag{1.2}$$

and the system of equations

$$\frac{\partial\theta_2}{\partial t} = \beta_{11}\frac{\partial^2\theta_2}{\partial x^2} + \beta_{12}\frac{\partial^2 U_2}{\partial x^2}, \quad y(t) < x < \infty, \tag{1.3}$$

$$\frac{\partial U_2}{\partial t} = \beta_{21}\frac{\partial^2 U_2}{\partial x^2} + \beta_{22}\frac{\partial^2\theta_2}{\partial x^2}, \quad y(t) < x < \infty \tag{1.4}$$

for the moist zone.

The following are the boundary and initial conditions:

$$\theta_1(0,\ t) = \theta_s = \text{const}, \quad U_1(0,\ t) = U_s = \text{const}, \tag{1.5}$$

$$\theta_2(x,\ 0) = \theta_s = \text{const}, \quad U_2(x,\ 0) = U_s = \text{const}. \tag{1.6}$$

At the moving boundary, i.e., at $x = y(t)$, we have equality in the temperature and water contents and also equalities in the water and heat fluxes, i.e.,

$$\theta_1(y(t),\ t) = \theta_2(y(t),\ t), \quad U_1(y(t),\ t) = U_2(y(t),\ t), \tag{1.7}$$

$$\left(\alpha_{21}\frac{\partial U_1}{\partial x} + \alpha_{22}\frac{\partial\theta_1}{\partial x}\right)_{x=y(t)} = \left(\beta_{21}\frac{\partial U_2}{\partial x} + \beta_{22}\frac{\partial\theta_2}{\partial x}\right)_{x=y(t)}, \tag{1.8}$$

$$\left[\lambda_1\frac{\partial\theta_1}{\partial x} - \nu_1\left(\alpha_{21}\frac{\partial U_1}{\partial x} + \alpha_{22}\frac{\partial\theta_1}{\partial x}\right)\right]_{x=y(t)} = \left[\lambda_2\frac{\partial\theta_2}{\partial x} - \nu_2\left(\beta_{21}\frac{\partial U_2}{\partial x} + \beta_{22}\frac{\partial\theta_2}{\partial x}\right)\right]_{x=y(t)}. \tag{1.9}$$

All the coefficients in the equation are considered as constant. The physical meanings of these coefficients are those given by Lykov [1] on the assumption that the heat and mass transfer involve vapor and water in the humid and evaporation zones.

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2. We seek particular solutions $\theta_1(x, t, p)$ and $U_1(x, t, p)$ to (1.1) and (1.2) in the form

$$\theta_1(x, t, p) = \theta_{11}(x, p)\exp(pt) \tag{2.1}$$

and

$$U_1(x, t, p) = U_{11}(x, p)\exp(pt). \tag{2.2}$$

We substitute (2.1) and (2.2) into (1.1) and (1.2) to get

$$p\theta_{11}(x, p) = \alpha_{11}\frac{d^2\theta_{11}}{dx^2} + \alpha_{12}\frac{d^2U_{11}}{dx^2} \tag{2.3}$$

and

$$pU_{11}(x, p) = \alpha_{21}\frac{d^2U_{11}}{dx^2} + \alpha_{22}\frac{d^2\theta_{11}}{dx^2}. \tag{2.4}$$

If we put

$$\theta_{11}(x, p) = -\alpha_{12}\frac{d^2}{dx^2}V(x, p), \tag{2.5}$$

and

$$U_{11}(x, p) = \left(\alpha_{11}\frac{d^2}{dx^2} - p\right)V(x, p), \tag{2.6}$$

where $V(x, p)$ is a new function to be determined, then (2.3) is satisfied identically, while from (2.4) we get

$$\left[\left(\alpha_{11}\frac{d^2}{dx^2} - p\right)\left(\alpha_{21}\frac{d^2}{dx^2} - p\right) - \alpha_{12}\alpha_{22}\frac{d^4}{dx^4}\right]V(x, p) = 0. \tag{2.7}$$

As (2.7) is a uniform equation of fourth order, the characteristic equation is

$$(\alpha_{11}\alpha_{21} - \alpha_{12}\alpha_{22})s^4 - p(\alpha_{11} + \alpha_{21})s^2 + p^2 = 0 \tag{2.8}$$

and this has four roots:

$$s_{1,2} = \pm r_1\sqrt{p} \text{ and } s_{3,4} = \pm r_2\sqrt{p}, \tag{2.9}$$

where

$$r_{1,2} = \sqrt{\frac{(\alpha_{11} + \alpha_{21}) \pm \sqrt{(\alpha_{11} + \alpha_{21})^2 - 4(\alpha_{11}\alpha_{21} - \alpha_{12}\alpha_{22})}}{2(\alpha_{11}\alpha_{21} - \alpha_{12}\alpha_{22})}} \tag{2.10}$$

If $\alpha_{11}\alpha_{21} - \alpha_{12}\alpha_{22} > 0$, all the roots are real; if on the other hand $\alpha_{11}\alpha_{21} - \alpha_{12}\alpha_{22} < 0$, then all the roots are imaginary. We consider the case of real roots, which corresponds to the physical conditions such that $\alpha_{11} > \alpha_{12}$ and $\alpha_{21} > \alpha_{22}$.

Then the general Eq. (2.7) will take the form

$$V(x, p) = C_1(p)\exp(r_1x\sqrt{p}) + C_2(p)\exp(-r_1x\sqrt{p}) + C_3(p)\exp(r_2x\sqrt{p}) + C_4(p)\exp(-r_2x\sqrt{p}), \tag{2.11}$$

where $C_i(p)$ are arbitrary functions of parameter p.

We substitute our result for $V(x, p)$ into (2.5) and (2.6) to get

$$\theta_{11}(x, p) = -\alpha_{12}r_1^2pF_1(x, p, r_1) - \alpha_{12}r_2^2pF_2(x, p, r_2) \tag{2.12}$$

and

$$U_{11}(x, p) = p(\alpha_{11}r_1^2 - 1)F_1(x, p, r_1) + p(\alpha_{11}r_2^2 - 1)F_2(x, p, r_2), \tag{2.13}$$

where

$$F_i(x, p, r_i) = C_j(p)\exp(r_ix\sqrt{p}) + C_{j+1}(p)\exp(-r_ix\sqrt{p})$$
$$(i = 1, 2, \text{ and } j = i \text{ for } i = 1 \text{ and } j = i + 1 \text{ for } i = 2). \tag{2.14}$$

Similarly we find the particular solutions $\theta_2(x, t, p) = \theta_{22}(x, p)\exp(pt)$ and $U_2(x, t, p) = U_{22}(x, p)\exp(pt)$; the functions $\theta_{22}(x, p)$ and $U_{22}(x, p)$ take the form

$$\theta_{22}(x, p) = -\beta_{12}k_1^2pF_3(x, p, k_1) - \beta_{12}k_2^2pF_4(x, p, k_2) \tag{2.15}$$

and

$$U_{22}(x, p) = p(\beta_{11}k_1^2 - 1)F_3(x, p, k_1) + p(\beta_{11}k_2^2 - 1)F_4(x, p, k_2), \tag{2.16}$$

where

$$F_i(x, p, k_i) = C_j(p)\exp(k_i x\sqrt{p}) + C_{j+1}(p)\exp(-k_i x\sqrt{p})$$
$$(i = 3,\ 4, \text{ and } j = i + 2 \text{ for } i = 3 \text{ and } j = i + 3 \text{ for } i = 4) \tag{2.17}$$

and

$$k_{1,2} = \sqrt{\frac{(\beta_{11} + \beta_{21}) \pm \sqrt{(\beta_{11} + \beta_{21})^2 - 4(\beta_{11}\beta_{21} - \beta_{12}\beta_{22})}}{2(\beta_{11}\beta_{21} - \beta_{12}\beta_{22})}} \tag{2.18}$$

provided that $\beta_{11}\beta_{21} - \beta_{12}\beta_{22} > 0$.

3. The general solutions to system (1.1)-(1.4) will be sought in the form of the following integrals:

$$\theta_l(x, t) = \frac{1}{2\pi i}\int_{\sigma - i\infty}^{\sigma + i\infty} \theta_{ll}(x, p)\exp(pt)\,dp \qquad (l = 1,\ 2), \tag{3.1}$$

and

$$U_l(x, t) = \frac{1}{2\pi i}\int_{\sigma - i\infty}^{\sigma + i\infty} U_{ll}(x, p)\exp(pt)\,dp \qquad (l = 1,\ 2). \tag{3.2}$$

Each of the integrals in (3.1) and (3.2) consists of the sum of four integrals; for instance, the integrals for $\theta_1(x, t)$ take the form

$$\theta_1(x, t) = -\alpha_{12}r_1^2\left[\frac{1}{2\pi i}\int_{\sigma - i\infty}^{\sigma + i\infty} \exp(pt + r_1 x\sqrt{p})\,pC_1(p)\,dp + \frac{1}{2\pi i}\int_{\sigma - i\infty}^{\sigma + i\infty} \exp(pt - r_1 x\sqrt{p})\,pC_2(p)\,dp\right] - \alpha_{12}r_2^2$$

$$\times\left[\frac{1}{2\pi i}\int_{\sigma - i\infty}^{\sigma + i\infty} \exp(pt + r_2 x\sqrt{p})\,pC_3(p)\,dp + \frac{1}{2\pi i}\int_{\sigma - i\infty}^{\sigma + i\infty} \exp(pt - r_2 x\sqrt{p})\,pC_4(p)\,dp\right]. \tag{3.3}$$

The calculations are considerably simplified if the arbitrary functions $C_j(p)$ for j = 1, 2, 3, 4, 5. 7 are put as

$$C_j(p) = \frac{C_j}{p^2}, \tag{3.4}$$

and for j = 6 and 8

$$C_j(p) = -\frac{C_j}{p^2}, \tag{3.5}$$

where C_j are arbitrary constants.

Also, (1.3) and (1.4) are homogeneous equations, so any constant will be a solution to them; bearing this in mind, we can transform the integrals containing the constants C_6 and C_8 in the form $C_j[1/p - \exp(-k_i x\sqrt{p})/p]$ with i = 1 for j = 6 and i = 2 for j = 8.

Then these values of $C_j(p)$ cause the integrals in (3.1) and (3.2) to take the values

$$\operatorname{erfc}\left(\pm\frac{r_i x}{2\sqrt{t}}\right) = 1 - \operatorname{erf}\left(\pm\frac{r_i x}{2\sqrt{t}}\right) = 1 - \frac{2}{\sqrt{\pi}}\int_0^{\pm\frac{r_i x}{2\sqrt{t}}} e^{-\alpha^2}\,d\alpha, \tag{3.6}$$

$$\operatorname{erfc}\left(\pm\frac{k_i x}{2\sqrt{t}}\right) = 1 - \operatorname{erf}\left(\pm\frac{k_i x}{2\sqrt{t}}\right) \tag{3.7}$$

and

$$\operatorname{erf}\left(\frac{k_i x}{2\sqrt{t}}\right) = \frac{2}{\sqrt{\pi}}\int_0^{\frac{k_i x}{2\sqrt{t}}} e^{-\alpha^2}\,d\alpha \qquad (i = 1,\ 2). \tag{3.8}$$

These values for the above integrals give the following form to the general solutions:

$$\theta_1(x,\ t) = -\alpha_{12}r_1^2\Phi_1\left(\frac{r_1x}{2\sqrt{t}}\right) - \alpha_{12}r_2^2\Phi_2\left(\frac{r_2x}{2\sqrt{t}}\right), \tag{3.9}$$

$$\theta_2(x,\ t) = -\beta_{12}k_1^2\Phi_3\left(\frac{k_1x}{2\sqrt{t}}\right) - \beta_{12}k_2^2\Phi_4\left(\frac{k_2x}{2\sqrt{t}}\right), \tag{3.10}$$

$$U_1(x,\ t) = (\alpha_{11}r_1^2 - 1)\,\Phi_1\left(\frac{r_1x}{2\sqrt{t}}\right) + (\alpha_{11}r_2^2 - 1)\,\Phi_2\left(\frac{r_2x}{2\sqrt{t}}\right) \tag{3.11}$$

and

$$U_2(x,\ t) = (\beta_{11}k_1^2 - 1)\,\Phi_3\left(\frac{k_1x}{2\sqrt{t}}\right) + (\beta_{11}k_2^2 - 1)\,\Phi_4\left(\frac{k_2x}{2\sqrt{t}}\right), \tag{3.12}$$

where

$$\Phi_i\left(\frac{r_ix}{2\sqrt{t}}\right) = C_j\,\mathrm{erfc}\left(-\frac{r_ix}{2\sqrt{t}}\right) + C_{j+1}\,\mathrm{erfc}\left(\frac{r_ix}{2\sqrt{t}}\right)$$

$(i = 1,\ 2,$ and $j = i$ for $i = 1$ and $j = i + 1$ for $i = 2),$

$$\Phi_l\left(\frac{k_ix}{2\sqrt{t}}\right) = C_n\,\mathrm{erfc}\left(-\frac{k_ix}{2\sqrt{t}}\right) + C_{n+1}\,\mathrm{erf}\left(\frac{k_ix}{2\sqrt{t}}\right)$$

$(l = 3,\ 4,$ and $i = 1$ and $n = l + 2$ for $l = 3$; for $l = 4$ $i = 2,$ and $n = l + 3).$

4. To find the arbitrary constants C_j from the boundary conditions of (1.5) we have two equations:

$$\alpha_{12}r_1^2(C_1 + C_2) + \alpha_{12}r_2^2(C_3 + C_4) = -\theta_s \tag{4.1}$$

and

$$(\alpha_{11}r_1^2 - 1)(C_1 + C_2) + (\alpha_{11}r_2^2 - 1)(C_3 + C_4) = -U_s; \tag{4.2}$$

and from the initial conditions of (1.6) we get two further equations:

$$\beta_{12}k_1^2(2C_5 + C_6) + \beta_{12}k_2^2(2C_7 + C_8) = -\theta_s \tag{4.3}$$

and

$$(\beta_{11}k_1^2 - 1)(2C_5 + C_6) + (\beta_{11}k_2^2 - 1)(2C_7 + C_8) = U_s. \tag{4.4}$$

We get the other four equations for the arbitrary constants from the four boundary conditions at the mobile evaporation boundary; however, substitution of (3.9)-(3.12) into (1.7)-(1.9) gives algebraic equations for the C_j only if the boundary moves in accordance with

$$x = y(t) = \beta\sqrt{t}, \tag{4.5}$$

where β is an arbitrary constant.

This law for the boundary gives us from (1.7)-(1.9) the following four equations:

$$-\alpha_{12}r_1^2\Phi_1\left(\frac{r_1\beta}{2}\right) - \alpha_{12}r_2^2\Phi_2\left(\frac{r_2\beta}{2}\right) + \beta_{12}k_1^2\Phi_3\left(\frac{k_1\beta}{2}\right) + \beta_{12}k_2^2\Phi_4\left(\frac{k_2\beta}{2}\right) = 0, \tag{4.6}$$

$$(\alpha_{11}r_1^2 - 1)\,\Phi_1\left(\frac{r_1\beta}{2}\right) + (\alpha_{11}r_2^2 - 1)\,\Phi_2\left(\frac{r_2\beta}{2}\right) - (\beta_{11}k_1^2 - 1)\,\Phi_3\left(\frac{k_1\beta}{2}\right) - (\beta_{11}k_2^2 - 1)\,\Phi_4\left(\frac{k_2\beta}{2}\right) = 0, \tag{4.7}$$

$$A_1(C_1 - C_2) + A_2(C_3 - C_4) - A_3(C_5 + C_6) - A_4(C_7 + C_8) = 0 \tag{4.8}$$

and

$$B_1(C_1 - C_2) + B_2(C_3 - C_4) + B_3(C_5 + C_6) + B_4(C_7 + C_8) = 0, \tag{4.9}$$

where

$$A_i = [r_i\alpha_{21}(\alpha_{11}r_i^2 - 1) - \alpha_{12}\alpha_{22}r_i^3]\exp\left(-\frac{r_i^2\beta^2}{4}\right) \quad (i = 1,\ 2);$$

$$A_{l+2} = [k_l\beta_{21}(\beta_{11}k_l^2 - 1) - \beta_{12}\beta_{22}k_l^3]\exp\left(-\frac{k_l^2\beta^2}{4}\right) \quad (l = 1,\ 2);$$

$$B_i = [r_i^3\alpha_{12}(\nu_1\alpha_{22} - \lambda_1) - \nu_1\alpha_{21}r_i(\alpha_{11}r_i^2 - 1)]\exp\left(-\frac{r_i^2\beta^2}{2}\right) \quad (i = 1,\ 2)$$

and

$$B_{l+2} = [k_l^3\beta_{12}(\lambda_2 - \nu_2\beta_{22}) + \nu_2\beta_{21}k_l(\beta_{11}k_l^2 - 1)] \exp\left(-\frac{k_l^2\beta^2}{4}\right) \quad (l = 1, 2).$$

Equations (4.1)-(4.4) and (4.6)-(4.9) constitute a system of eight inhomogeneous equations, which are to be solved for the C_j. These C_j are dependent also on β; we get the equation for β from the condition that $U_1(\beta\sqrt{t}, t) = U_c$ at the mobile boundary.

We substitute $x = \beta\sqrt{t}$ into (3.11) and use $U_1(\beta\sqrt{t}, t) = U_c$ to get

$$U_c = (\alpha_{11}r_1^2 - 1)\,\Phi_1\left(\frac{r_1\beta}{2}\right) + (\alpha_{11}r_2^2 - 1)\,\Phi_2\left(\frac{r_2\beta}{2}\right) \tag{4.10}$$

to determine β.

NOTATION

$\theta_1(x, t)$ and $\theta_2(x, t)$	are the temperatures in the evaporation zone and in the humid zone;
$U_1(x, t)$ and $U_2(x, t)$	are the humidities in the evaporation zone and in the humid zone;
$y(t) = \beta\sqrt{t}$	is the position of the evaporation boundary.

LITERATURE CITED

1. A. V. Lykov, Theory of Drying [in Russian], Énergiya (1968).